Associative Omega-products of Traces*

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Abstract

The notion of associative infinite product is applied to traces, resulting in an alternative approach to introducing infinite traces. Four different versions of product are explored, two of them identical to known definitions of infinite trace.

1 Introduction

Extending binary operation to an infinite sequence of operands is not a new idea. A classical example is the infinite series, which is such an extension of "+". Newer examples are infinite concatenation of words and concatenation product of an infinite sequence of languages. Some of these extensions are associative, that is, the result does not change if the factors are grouped by parentheses. Some are not, like the infinite series in the domain of all real numbers. For a long time, associativity has been exploited in a rather informal way. But in recent years, the research connecting automata, semigroups, and infinite-word languages required a more formal treatment of infinite associativity. The infinite products appearing in that context do not have the intuitive form of being the limit of longer and longer finite products; thus the need for a precise treatment.

It seems that the first formal treatment of infinite product (shortly: *ω-product*) was published in [10]. Slightly before, the present author proposed a set of axioms for an associative ω-product in a Dagstuhl Seminar lecture, of which only an abstract [12] was published. Being applied to finite automata, the infinite product was mainly studied for finite semigroups. An extensive review can be found in [11]. A recent paper [13] by the present author is a general study of associative ω-products for arbitrary semigroups.

The present paper applies some results from [13] to the semigroup of finite traces. Traces were introduced in [7] to describe behavior of concurrent systems. A rich theory has been developed since then. The reader is referred to [9] for a survey. The traces, as originally defined, described finite behavior. In order to describe systems that never stop, it was necessary to consider infinite traces. Infinite traces were first introduced in [8]; then, in a different form, independently in [2] and [5]. A complete presentation can be found in [4].

Intuitively, the result of an infinite product of finite traces should be an infinite trace. We obtain in this way an alternative approach to defining infinite traces. Unfortunately, associativity alone does not uniquely define the result of an infinite product. The result can be freely chosen in a number of different ways, some more meaningful than other. We explore four of possible choices, one of them matching the definition from [8] and one that from [2, 5].

We begin, in Section 2, by recalling the necessary definitions and results from [13]. We discuss associative ω-products in the general setting of an arbitrary semigroup. In Section 3, we recall basic definitions and facts concerning traces. In Section 4, we apply the theory from Section 2 to the semigroup of finite non-null traces. We identify the "similarity classes" of that semigroup – a basic result needed to construct associative ω-products. We apply that result in Section 5 to suggest four such products. In Section 6, we discuss possible extensions of these products to null traces. Section 7 contains some final remarks.

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2 Associative ω-products

The set of all natural numbers (positive integers) is in the following denoted by $\mathbb{N}$. A sequence $\mathbf{x}$ of elements of a set S is a mapping $\mathbf{x} : \mathbb{N} \to S$. It is visualized as a linear arrangement of elements $\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \dots$. The set of all sequences of elements of S is denoted by $S^{\mathbb{N}}$. The sequence $s, \mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \dots$ obtained by adding $s \in S$ in front of sequence $\mathbf{x} \in S^{\mathbb{N}}$ is denoted by $s, \mathbf{x}$.

A *semigroup* is a pair $(S, \cdot)$ where S is a set, and $\cdot$ is an associative operation on S, referred to as the *semigroup product*.

An ω*-product* on $(S, \cdot)$ is a mapping π from $S^{\mathbb{N}}$ to some set V of *values*. These values may belong to S (as in the case of infinite series), or be outside S (as in the case of infinite concatenation of words). As indicated before, $\pi(s_1, s_2, s_3, \dots)$ need not be any kind of limit of partial products $s_1 \cdot s_2 \cdot \ldots \cdot s_n$.

It is convenient to write $\pi(s_1, s_2, s_3, \dots)$ informally as $s_1 \cdot s_2 \cdot s_3 \cdot \ldots$. In this form, symbol $\cdot$ denotes the ω-product, not an operation on two neighboring factors. Informally speaking, we mean that π is "associative" if it satisfies identities of this kind:

$$s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \ldots = (s_1 \cdot s_2 \cdot s_3) \cdot (s_4 \cdot s_5 \cdot s_6) \cdot \ldots , \tag{1}$$

$$s_1 \cdot s_2 \cdot s_3 \cdot \ldots = (s_1 \cdot \ldots \cdot s_n) \cdot (s_{n+1} \cdot s_{n+2} \cdot s_{n+3} \cdot \ldots) . \tag{2}$$

In (1), we understand the dot within parentheses to mean the semigroup product, and outside parentheses to mean the ω-product. In (2), the dot within the first pair of parentheses denotes the semigroup product; within the second pair, it denotes the ω-product. The dot in the middle stands for an operation $S \times V \to V$ defined by $s \cdot \pi(\mathbf{x}) = \pi(s, \mathbf{x})$ and called the *mixed product*.

In order to express (1) more precisely, we define, for $\mathbf{x} = s_1, s_2, s_3, \ldots \in S^{\mathbb{N}}$ and ascending $\mathbf{n} = n_1, n_2, n_3 \ldots \in \mathbb{N}^{\mathbb{N}}$, the sequence $\mathbf{x}|\mathbf{n}$ as:

$$\mathbf{x}|\mathbf{n} = (s_1 \cdot \ldots \cdot s_{n_1}),\ (s_{n_1+1} \cdot \ldots \cdot s_{n_2}),\ (s_{n_2+1} \cdot \ldots \cdot s_{n_3}),\ \ldots . \tag{3}$$

For example: $(s_1, s_2, s_3, \ldots)|(1, 3, 4, 6, \ldots) = (s_1),\ (s_2 \cdot s_3),\ (s_4),\ (s_5 \cdot s_6),\ \ldots$.

Formally, we say that ω-product $\pi : S^{\mathbb{N}} \to V$ is *associative* if it has these three properties:

(A1) $\pi(\mathbf{x}) = \pi(\mathbf{x}|\mathbf{n})$ for all $\mathbf{x} \in S^{\mathbb{N}}$ and ascending $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$.

(A2) $\pi(\mathbf{x}) = \pi(\mathbf{y}) \Rightarrow \pi(s, \mathbf{x}) = \pi(s, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$ and $s \in S$.

(A3) $\pi(s, \mathbf{x}) = s \cdot \pi(\mathbf{x})$ for all $s \in S$ and $\mathbf{x} \in S^{\mathbb{N}}$ such that $\pi(\mathbf{x}) \in S$.

Property (A1) states validity of all equations of form (1); (A2) ensures that mixed product is uniquely defined, and (A3) ensures that (2) is not ambiguous.

Our purpose is to construct associative ω-products by choosing values of $\pi(\mathbf{x})$ for different sequences $\mathbf{x} \in S^{\mathbb{N}}$. According to (A1) we must have $\pi(\mathbf{x}) = \pi(\mathbf{y})$ whenever $\mathbf{x}$ and $\mathbf{y}$ can be transformed into each other in a finite number of steps using the operation defined by (3). In the following, such sequences $\mathbf{x}, \mathbf{y}$ are called *similar*, written $\mathbf{x} \sim \mathbf{y}$. More precisely, $\mathbf{x} \sim \mathbf{y}$ means that there exist sequences $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ in $S^{\mathbb{N}}$ and ascending sequences $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{k-1}$ of natural numbers such that $\mathbf{x} = \mathbf{z}_1$, $\mathbf{y} = \mathbf{z}_k$, and either $\mathbf{x}_i|\mathbf{n}_i = \mathbf{x}_{i+1}$ or $\mathbf{x}_i = \mathbf{x}_{i+1}|\mathbf{n}_i$ for $1 \le i < k$. The relation $\sim$ is obviously an equivalence in $S^{\mathbb{N}}$; its equivalence classes are referred to as the *similarity classes* of $(S, \cdot)$.

An ω-product satisfying (A1) is obtained by assigning an arbitrary value to each similarity class q and using it as $\pi(\mathbf{x})$ for each $\mathbf{x} \in q$. If values thus assigned to different q are distinct, meaning $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \pi(\mathbf{x}) = \pi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$, π satisfies both (A1) and (A2). If, in addition, none of the assigned values is in S, π satisfies all of (A1)–(A3).

3 Words and traces

An *alphabet* A is a finite nonempty set of *letters.* A *word* is a finite or infinite string of letters. The number of letters in a finite word is called its *length.* The word of length 0 (string of no letters) is called the *null word* and is denoted by ε. Words are otherwise denoted by (possibly subscripted) letters u, v, x, y, z, with u and v reserved for infinite words. The set of finite words, including ε, is denoted by A^*, the set of infinite words by A^ω, and the set of all words by A^∞.

Concatenation of words $x \in A^*$ and $y \in A^\infty$ is the word obtained by appending y at the end of x. It is denoted by xy. The infinite word obtained by joining an infinite sequence of finite non-null words $x_1, x_2, x_3, \ldots$ one after another is denoted by $x_1x_2x_3 \ldots$.

Word $y \in A^*$ is a *prefix* of word $x \in A^\infty$, denoted $y \leq x$, if $x = yz$ for some $z \in A^\infty$.

Traces are equivalence classes of certain congruence $\equiv$ on the semigroup $(A^*, \cdot)$, where $\cdot$ is concatenation of words. This congruence is defined by *independence relation* $I \subseteq A \times A$ as the smallest congruence $\equiv$ such that $(a, b) \in I \Rightarrow ab \equiv ba$. The independence relation and the congruence defined by it remain fixed for the rest of the paper. The set of all traces is denoted by T. The trace containing word $x \in A^*$ is denoted by $[x]$. All words in $[x]$ are certain permutations of letters in the word x, and thus all have the same length. This is the *length* of $[x]$.

Relation $\equiv$ being a congruence means that there exists quotient operation $\cdot$, the *trace product*, defined by $[x] \cdot [y] = [xy]$ for $[x], [y] \in T$. The trace product is obviously associative. It satisfies left-cancellation law: $[x] \cdot [y] = [x] \cdot [z] \ \Rightarrow\ [y] = [z] \quad \text{for} \quad [x], [y], [z] \in T$.

Trace $[y] \in T$ is a *prefix* of trace $[x] \in T$, denoted $[y] \leq [x]$, if $[x] = [y] \cdot [z]$ for some $[z] \in T$. We note that for $x, y \in A^*$, $y \leq x$ implies $[y] \leq [x]$, and $[y] \leq [x]$ implies $y \leq z$ for some $z \in [x]$.

The prefix relations $\leq$ on A^* and T are partial orders on the respective sets. Because we consider only finite word prefixes, $\leq$ is not a partial order on all of A^∞, but it is still transitive whenever defined. We apply few standard concepts from order theory (see, for example, [1]) to prefix relations on A^∞ and T. They apply generally to a set D with a transitive relation $\leq$. For a subset $P \subseteq D$, we define $\downarrow P = \{r \in D \,|\, r \leq p \ \text{ for some } \ p \in P\}$, and abbreviate $\downarrow\{p\}$ as $\downarrow p$. Subset $P \subseteq D$ is a *lower set* if $P = \downarrow P$; it is *directed* if for each $r, s \in P$ exists $p \in P$ such that $r \leq p$ and $s \leq p$. A directed lower set is an *ideal.* An element of P is *maximal* if it is not a prefix of any other element of P.

The use of common notation should not cause ambiguities, as traces are always written in the form $[x]$ with $x \in A^*$, easily distinguishable from words. The only possible exception is the set $\downarrow[x]$. Here, $[x]$ is always treated as a member of T, not as set of words; $\downarrow[x]$ is thus always a set of traces.

4 Similar sequences of traces

As it will be shown in Section 6, the null trace $[\varepsilon]$ causes complications and gives rise to special cases. Therefore we shall only consider ω-products on the semigroup $\mathbf{T} = (T_+, \cdot)$ where $T_+ = T - [\varepsilon]$. We start by identifying the similarity classes of $\mathbf{T}$. For this purpose, we define the *signature* of a sequence $[x_1], [x_2], [x_3], \ldots \in T_+^{\mathbb{N}}$ as:

$$sign([x_1], [x_2], [x_3], \ldots) = \bigcup_{i \geq 1} \downarrow[x_1x_2 \ldots x_i] \ . \tag{4}$$

Let now $\mathbf{x} = [x_1], [x_2], [x_3], \ldots$ and $\mathbf{y} = [y_1], [y_2], [y_3], \ldots$ be two arbitrary sequences from $T_+^{\mathbb{N}}$.

Lemma 1. *For any ascending* $\mathbf{n} = n_1, n_2, n_3, \ldots \in \mathbb{N}^{\mathbb{N}}$, *we have* $\mathbf{y} = \mathbf{x}|\mathbf{n}$ *if and only if* $[y_1 \ldots y_i] = [x_1 \ldots x_{n_i}]$ *for* $i \geq 1$.

Proof. Suppose $\mathbf{y} = \mathbf{x}|\mathbf{n}$. From (3) follows immediately $[y_1 \ldots y_i] = [x_1 \ldots x_{n_i}]$ for all $i \geq 1$.
Suppose now that $[y_1 \ldots y_i] = [x_1 \ldots x_{n_i}]$ for all $i \geq 1$.
For each $i > 1$ we have $[y_1 \ldots y_{i-1}] \cdot [y_i] = [x_1 \ldots x_{n_{i-1}}] \cdot [x_{n_{i-1}+1} \ldots x_{n_i}]$.
But $[y_1 \ldots y_{n-1}] = [x_1 \ldots x_{n_{i-1}}]$; thus, by left-cancellation, $[y_i] = [x_{n_{i-1}+1} \ldots x_{n_i}]$. In addition, we have $[y_1] = [x_1 \ldots x_{n_1}]$, showing that $\mathbf{y} = \mathbf{x}|\mathbf{n}$. □

Lemma 2. *For any ascending* $\mathbf{n} = n_1, n_2, n_3, \ldots \in \mathbb{N}^{\mathbb{N}}$, $\mathbf{y} = \mathbf{x}|\mathbf{n}$ *implies* $sign(\mathbf{x}) = sign(\mathbf{y})$.

Proof. Suppose $\mathbf{y} = \mathbf{x}|\mathbf{n}$. By Lemma 1, we have $[y_1 \dots y_i] = [x_1 \dots x_{n_i}]$ for $i \geq 1$.
Take any $[z] \in \mathit{sign}(\mathbf{x})$. That means $[z] \leq [x_1 \dots x_j]$ for some $j \geq 1$. As $\mathbf{n}$ is ascending, there exists i such that $n_i > j$. We have then $[z] \leq [x_1 \dots x_j \dots x_{n_i}] = [y_1 \dots y_i]$, so $[z] \in \mathit{sign}(\mathbf{y})$.
Take now any $[z] \in \mathit{sign}(\mathbf{y})$; that means $[z] \leq [y_1 \dots y_i]$ for some $i \geq 1$. But $[y_1 \dots y_i] = [x_1 \dots x_{n_i}]$, so $[z] \in \mathit{sign}(\mathbf{x})$. $\square$

Lemma 3. *If $\mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$, there exist sequence $\mathbf{z} = [z_1], [z_2], [z_3], \dots \in T_+^{\mathbb{N}}$ and ascending sequences $\mathbf{n} = n_1, n_2, n_3, \dots \in \mathbb{N}^{\mathbb{N}}$, $\mathbf{m} = m_1, m_2, m_3, \dots \in \mathbb{N}^{\mathbb{N}}$ such that $[x_1 \dots x_{n_i}] = [z_1 \dots z_{2i-1}]$ and $[y_1 \dots y_{m_i}] = [z_1 \dots z_{2i}]$ for $i \geq 1$.*

Proof. Suppose $\mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$. The required sequences can be constructed as follows.

Take any $n_1 \geq 1$ and define $[z_1] = [x_1 \dots x_{n_1}]$. Clearly, $[z_1] \in \mathit{sign}(\mathbf{x})$. As $\mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$, we have $[z_1] \leq [y_1 \dots y_j]$ for some $j \geq 1$. Define $m_1 = j + 1$. We have then $[y_1 \dots y_j y_{m_1}] = [z_1] \cdot [z_2]$ for some $[z_2] \neq [\varepsilon]$. The $[z_1]$, $[z_2]$, n_1, m_1 thus obtained are the required elements of $\mathbf{z}, \mathbf{n}, \mathbf{m}$ for $i = 1$.

Suppose the required elements of $\mathbf{n}, \mathbf{m}$, and $\mathbf{z}$ have been constructed up to some $i \geq 1$. In particular, we have $[z_1], \dots, [z_{2i}]$ and m_i such that $[z_1 \dots z_{2i}] = [y_1 \dots y_{m_i}]$.
Clearly, $[z_1 \dots z_{2i}] \in \mathit{sign}(\mathbf{y})$. As $\mathit{sign}(\mathbf{y}) = \mathit{sign}(\mathbf{x})$, we have $[z_1 \dots z_{2i}] \leq [x_1 \dots x_k]$ for some $k \geq 1$. Define n_{i+1} to be the greater of $k + 1$ and $n_i + 1$. We have then $[x_1 \dots x_k \dots x_{n_{i+1}}] = [z_1 \dots z_{2i}] \cdot [z_{2i+1}]$ for some $[z_{2i+1}] \neq [\varepsilon]$.
Clearly, $[z_1 \dots z_{2i+1}] \in \mathit{sign}(\mathbf{x})$. As $\mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$, we have $[z_1 \dots z_{2i+1}] \leq [y_1 \dots y_j]$ for some $j \geq 1$. Define m_{i+1} to be the greater of $j+1$ and m_i+1. We have then $[y_1 \dots y_j \dots y_{m_{i+1}}] = [z_1 \dots z_{2i+1}] \cdot [z_{2i+2}]$ for some $[z_{2i+2}] \neq [\varepsilon]$. The $[z_{2i+1}]$, $[z_{2i+2}]$, n_{i+1}, m_{i+1} thus obtained are the required elements of $\mathbf{z}, \mathbf{n}, \mathbf{m}$ for $i + 1$. $\square$

Proposition 1. $\mathbf{x} \sim \mathbf{y}$ *if and only if* $\mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$.

Proof. Suppose $\mathbf{x} \sim \mathbf{y}$. Let $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n$ be the sequences in the definition of $\mathbf{x} \sim \mathbf{y}$. By Lemma 2, all these sequences have identical signatures.

Suppose now that $\mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$. Let $\mathbf{n}, \mathbf{m}, \mathbf{z}$ be as stated by Lemma 3. Define $\mathbf{p} = 1, 3, 5, \dots$ and $\mathbf{q} = 2, 4, 6, \dots$. By Lemma 1, we have $\mathbf{x}|\mathbf{n} = \mathbf{z}|\mathbf{p}$ and $\mathbf{z}|\mathbf{q} = \mathbf{y}|\mathbf{m}$, showing that $\mathbf{x} \sim \mathbf{z} \sim \mathbf{y}$. $\square$

To close this section, we note an important special case:

Proposition 2. *Any sequences $\mathbf{x} = [x_1], [x_2], [x_3], \dots \in T_+^{\mathbb{N}}$ and $\mathbf{y} = [y_1], [y_2], [y_3], \dots \in T_+^{\mathbb{N}}$ such that $x_1 x_2 x_3 \dots = y_1 y_2 y_3 \dots$ are similar.*

Proof. Equality of infinite words $x_1 x_2 x_3 \dots$ and $y_1 y_2 y_3 \dots$ means that they are the same sequence $a_1 a_2 a_3 \dots$ of letters a_i. There exist sequences $\mathbf{n} = n_1, n_2, n_3, \dots \in \mathbb{N}^{\mathbb{N}}$ and $\mathbf{m} = m_1, m_2, m_3, \dots \in \mathbb{N}^{\mathbb{N}}$ such that:

$$
\begin{aligned}
&x_1 = a_1 \dots a_{n_1}, \quad x_2 = a_{n_1+1} \dots a_{n_2}, \quad x_3 = a_{n_2+1} \dots a_{n_3}, \quad \dots, \\
&y_1 = a_1 \dots a_{m_1}, \quad y_2 = a_{m_1+1} \dots a_{m_2}, \quad y_3 = a_{m_2+1} \dots a_{m_3}, \quad \dots .
\end{aligned}
$$

Let $\mathbf{a} = [a_1], [a_2], [a_3], \dots$. One can easily see that $\mathbf{a}|\mathbf{n} = \mathbf{x}$ and $\mathbf{a}|\mathbf{m} = \mathbf{y}$, so $\mathbf{x} \sim \mathbf{y}$. $\square$

5 Associative ω-products of traces

As defined in Section 2, an associative ω-product on $\mathbf{T} = (T_+, \cdot)$ is a mapping $\pi : T_+^{\mathbb{N}} \to V$ that satisfies (A1)–(A3). Following the convention introduced there, we can write $\pi([x_1], [x_2], [x_3], \dots)$ informally as $[x_1] \cdot [x_2] \cdot [x_3] \cdot \dots$. From Proposition 2 follows that an associative product $[x_1] \cdot [x_2] \cdot [x_3] \cdot \dots$ is uniquely defined by the word $x_1 x_2 x_3 \dots$, so it can be denoted by $[x_1 x_2 x_3 \dots]$. We can thus write $[x_1] \cdot [x_2] \cdot [x_3] \cdot \dots = [x_1 x_2 x_3 \dots]$ in analogy to $[x] \cdot [y] = [xy]$. With such notation, associativity of π allows identities like these:

$$
\begin{aligned}
&[x_1 x_2 x_3 \dots] = [x_1 x_2] \cdot [x_2 x_3 x_4] \cdot [x_5 x_6] \cdot \dots, \\
&[x_1 x_2 x_3 \dots] = [x_1 x_2] \cdot [x_3 x_4 \dots].
\end{aligned}
$$

These remarks apply to any associative ω-product of traces. There are many possible such products, and we are going to explore some of them. As indicated in Section 2, one way to obtain an associative ω-product is by assigning values to π so that $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \pi(\mathbf{x}) = \pi(\mathbf{y})$ and $\pi(\mathbf{x}) \notin T_+$ for $\mathbf{x}, \mathbf{y} \in T_+^{\mathbb{N}}$. We consider four different such assignments, resulting in four different versions of associative ω-product.

5.1 Version 1: set of traces as value

According to Proposition 1, similarity classes of $\mathbf{T}$ consist of sequences $\mathbf{x}$ having the same signature. The simplest way of assigning distinct values to different similarity classes is to use that signature as the value. Following this idea we define, for $\mathbf{x} \in T_+^{\mathbb{N}}$:

$$\pi_1(\mathbf{x}) = \mathit{sign}(\mathbf{x}) \,. \tag{5}$$

The mapping $\pi_1 : T_+^{\mathbb{N}} \to 2^T$ is an ω-product on $\mathbf{T}$. It satisfies $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \pi_1(\mathbf{x}) = \pi_1(\mathbf{y})$ by Proposition 1. As $\mathit{sign}(\mathbf{x})$ is a set of traces, we have $\pi_1(\mathbf{x}) \notin T_+$, and π_1 is associative.

Intuitively, the product of infinitely many traces should be an infinite trace. And indeed, when infinite trace was introduced for the first time by Mazurkiewicz in [8], it was defined as a set of traces; more precisely, as an infinite ideal of T. We proceed to show that values of our infinite product are exactly the infinite traces in that sense.

Proposition 3. *The values of $\pi_1(\mathbf{x})$ for $\mathbf{x} \in T_+^{\mathbb{N}}$ are exactly the infinite ideals of T.*

Proof. (1) For each $\mathbf{x} \in T_+^{\mathbb{N}}$, $\mathit{sign}(\mathbf{x})$ is an infinite ideal of T.
Take any $\mathbf{x} = [x_1], [x_2], [x_3], \ldots \in T_+^{\mathbb{N}}$, and denote $P = \mathit{sign}(\mathbf{x})$. For each $i \geq 1$, P has a member of length i or greater, namely $[x_1 \ldots x_i]$; it is thus infinite.
Take now any $[x] \in P$ and $[y] \leq [x]$. We have $[y] \leq [x] \leq [x_1 \ldots x_i]$ for some $i \geq 1$, so $[y] \leq [x_1 \ldots x_i]$ and $[y] \in P$. This shows that P is a lower set.
Finally, take any $[x], [y] \in P$. We have $[x] \leq [x_1 \ldots x_i]$ for some $i \geq 1$, and $[y] \leq [x_1 \ldots x_j]$ for some $j \geq 1$. Let k be any integer greater than i and j. Denote $[z] = [x_1 \ldots x_i \ldots x_k] = [x_1 \ldots x_j \ldots x_k]$ Clearly, $[z] \in P$, $[x] \leq [z]$ and $[y] \leq [z]$. This shows that P is directed.

(2) Each infinite ideal of T is the signature of some sequence $\mathbf{x} \in T_+^{\mathbb{N}}$.
Let P be any infinite ideal of T. Denote by P_n the set of all members of P having length n or less. Because the alphabet A is finite, so is P_n. From P being an infinite lower set follows that $P_n \neq \varnothing$ for all $n \geq 0$. Using the fact of P being directed, one can find for every $n \geq 0$ an element $[y_n] \in P - P_n$ such that $P_n \subset \downarrow [y_n]$. Using this fact, one can choose $n_1, n_2, n_3, \ldots$ such that $[y_{n_1}] \leq [y_{n_2}] \leq [y_{n_3}] \leq \ldots$. As one can easily see, there exists $\mathbf{x} = [x_1], [x_2], [x_3], \ldots \in T_+^{\mathbb{N}}$ such that $[x_1 \ldots x_i] = [y_{n_i}]$ for all $i \geq 1$.
To complete the proof of (2), we verify that $P = \mathit{sign}(\mathbf{x})$. Take first any $[z] \in P$. We have $[z] \leq [y_{n_i}]$ for any n_i greater than the length of $[z]$. But $[y_{n_i}] = [x_1 \ldots x_i]$, so $[z] \in \mathit{sign}(\mathbf{x})$.
Take now any $[z] \in \mathit{sign}(\mathbf{x})$. We have $[z] \leq [x_1 \ldots x_i] = [y_{n_i}]$ for some i. But $[y_{n_i}] \in P$, so $[z] \in P$ by P being a lower set. $\square$

5.2 Version 2: set of finite words as value

An infinite sequence $\mathbf{x}$ of traces is presumably intended to represent some infinite behavior. The traces that are members of $\mathit{sign}(\mathbf{x})$ would then represent possible initial sequences of events. It may be more convenient to represent this initial behavior by a set of words rather than set of traces. Following this idea, we define the *word signature* of $\mathbf{x} \in T_+^{\mathbb{N}}$ as the union of members of $\mathit{sign}(\mathbf{x})$:

$$\mathit{wsign}(\mathbf{x}) = \{x \in A^* \mid [x] \in \mathit{sign}(\mathbf{x})\} \,, \tag{6}$$

and use it as the value of new ω-product:

$$\pi_2(\mathbf{x}) = \mathit{wsign}(\mathbf{x}) \,. \tag{7}$$

The mapping $\pi_2 : T_+^{\mathbb{N}} \to 2^{A^*}$ is an ω-product on $\mathbf{T}$. One can easily see that $\mathit{sign}(\mathbf{x}) = \mathit{wsign}(\mathbf{x})/\!\equiv$, so $\mathit{wsign}(\mathbf{x}) = \mathit{wsign}(\mathbf{y}) \Leftrightarrow \mathit{sign}(\mathbf{x}) = \mathit{sign}(\mathbf{y})$. From this follows $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \pi_2(\mathbf{x}) = \pi_2(\mathbf{y})$. As $\mathit{wsign}(\mathbf{x})$ is an infinite set of words, we have $\pi_2(\mathbf{x}) \notin T_+$, and π_2 is associative.

We can now define infinite traces to be the values of π_2. Unfortunately, they do not seem to have a nice characterization such as stated by Proposition 3 for values of π_1. In particular, they are, as a rule, not directed sets. For further use, we note the following property:

Proposition 4. *For each $\mathbf{x} \in T_+^{\mathbb{N}}$, $wsign(\mathbf{x})$ is a lower set without maximal elements.*

Proof. Consider any $\mathbf{x} = [x_1], [x_2], [x_3], \ldots \in T_+^{\mathbb{N}}$.
Take any $x \in wsign(\mathbf{x})$. By (6), $[x]$ belongs to $sign(\mathbf{x})$. Consider any $y \in A^*$ such that $y \leq x$. This implies $[y] \leq [x]$. According to Proposition 3, $sign(\mathbf{x})$ is a lower set, so $[y] \leq [x]$ implies $[y] \in sign(\mathbf{x})$. By (6), we have $y \in wsign(\mathbf{x})$, showing that $wsign(\mathbf{x})$ is a lower set.
According to (4), $[x] \in sign(\mathbf{x})$ means $[x] \leq [x_1 \ldots x_i] \leq [x_1 \ldots x_i x_{i+1}]$ for some $i \geq 1$. From $[x] \leq [x_1 \ldots x_i x_{i+1}]$ follows $x \leq z$ for some $z \in [x_1 \ldots x_i x_{i+1}]$. As $[x] \leq [x_1 \ldots x_i]$ and $x_{i+1} \neq \varepsilon$, z is longer than, and thus different from, x. Clearly, $[x_1 \ldots x_i x_{i+1}] \in sign(\mathbf{x})$, so $z \in wsign(\mathbf{x})$ by (6). Hence, x is a prefix of another word in $wsign(\mathbf{x})$ and thus not maximal. $\square$

5.3 Version 3: set of infinite words as value

A set of finite words with properties stated by Proposition 4 can be uniquely represented by the set of its least upper bounds. Following this idea we define, for $\mathbf{x} \in T_+^{\mathbb{N}}$:

$$\pi_3(\mathbf{x}) = \{u \in A^\omega \mid {\downarrow} u \subseteq wsign(\mathbf{x})\}\,. \tag{8}$$

The mapping $\pi_3 : T_+^{\mathbb{N}} \to 2^{A^\omega}$ is an ω-product on $\mathbf{T}$. We check that $\pi_3(\mathbf{x})$ indeed uniquely identifies $wsign(\mathbf{x})$:

Proposition 5. *For any $\mathbf{x} \in T_+^{\mathbb{N}}$, ${\downarrow}(\pi_3(\mathbf{x})) = wsign(\mathbf{x})$.*

Proof. Consider any $\mathbf{x} = [x_1], [x_2], [x_3], \ldots \in T_+^{\mathbb{N}}$.
Take any $x \in {\downarrow}(\pi_3(\mathbf{x}))$, that is, $x \in {\downarrow} u$ for some $u \in \pi_3(\mathbf{x})$. According to (8), $x \in wsign(\mathbf{x})$.
Take now any $x \in wsign(\mathbf{x})$. According to Proposition 4, $wsign(\mathbf{x})$ does not have maximal elements. That means, $wsign(\mathbf{x})$ contains an infinite ascending chain $x \leq z_1 \leq z_2 \leq z_3 \leq \ldots$. There exists unique infinite word u having all these words as prefixes. According to Proposition 4, $wsign(\mathbf{x})$ is a lower set. That means it contains all prefixes of all z_i for $i \geq 1$, and thus all prefixes of u. According to (8), $u \in \pi_3(\mathbf{x})$. As $x \leq u$, we have $x \in {\downarrow}(\pi_3(\mathbf{x}))$. $\square$

It follows that $\pi_3(\mathbf{x}) = \pi_3(\mathbf{y}) \Leftrightarrow wsign(\mathbf{x}) = wsign(\mathbf{y}) \Leftrightarrow \mathbf{x} \sim \mathbf{y}$. As $\pi_3(\mathbf{x})$ consists of infinite words, we have $\pi_3(\mathbf{x}) \notin T_+$, and π_3 is associative.

We could regard the values of π_3 as infinite traces. Because $\pi_3(\mathbf{x})$ identifies the sequence $\mathbf{x}$ up to similarity, it uniquely identifies the behavior represented by $\pi_1(\mathbf{x})$ and $\pi_2(\mathbf{x})$. But, one cannot interpret the infinite words in $\pi_3(\mathbf{x})$ as possible sequences of events belonging to that behavior. As an example, suppose $A = \{a, b, c\}$ with $(a, b) \in I$. Consider the sequence $\mathbf{x} = [ab], [ab], [ab], \ldots$. Each partial product $[ab] \cdot \ldots \cdot [ab]$ consists of all permutations of an equal number of a's and b's. But, $wsign(\mathbf{x})$ contains words a^n for any $n \geq 1$. These are all prefixes of a^ω, so $a^\omega \in \pi_3(\mathbf{x})$. This does not agree well with the intuition of $[ab] \cdot [ab] \cdot [ab] \cdot \ldots$ describing infinite behavior represented by $\mathbf{x}$; we would expect it to contain only words with infinitely many a's and b's.

One can obtain a better representation by choosing a different set of infinite words. The point is that different subsets of A^ω may define the same set $wsign(\mathbf{x})$.

5.4 Version 4: another set of infinite words as value

Let us define the signature of an infinite word $u \in A^\omega$ as:

$$sign(u) = \bigcup_{x \leq u} {\downarrow}[x]\,. \tag{9}$$

The signatures of words and sequences are closely related:

Proposition 6. *$sign([x_1], [x_2], [x_3], \ldots) = sign(x_1 x_2 x_3 \ldots)$ for any $[x_1], [x_2], [x_3], \ldots \in T_+^{\mathbb{N}}$.*

Proof. The infinite word $x_1x_2x_3\ldots$ is a string of letters, that is, $x_1x_2x_3\ldots = a_1a_2a_3\ldots$ where $a_i \in A$ for $i \geq 1$. According to in (9), $sign(x_1x_2x_3\ldots)$ is the union of $\downarrow[a_1\ldots a_n]$ for $n \geq 0$. This is identical to $sign([a_1],[a_2],[a_3],\ldots)$. By Proposition 2, $[x_1],[x_2],[x_3],\ldots \sim [a_1],[a_2],[a_3],\ldots$. The stated result follows from Proposition 1. □

We recall that each similarity class of $\mathbf{T}$ consists of sequences having the same signature. According to Proposition 6, the set of words having exactly that signature is nonempty for each similarity class. We can use this set as value of an ω-product. Following this idea we define, for $\mathbf{x} \in T_+^{\mathbb{N}}$:

$$\pi_4(\mathbf{x}) = \{u \in A^\omega \mid sign(u) = sign(\mathbf{x})\}\,. \tag{10}$$

The mapping $\pi_4 : T_+^{\mathbb{N}} \to 2^{A^\omega}$ is an ω-product on $\mathbf{T}$. Different signatures define different sets of words, so $\pi_4(\mathbf{x}) = \pi_4(\mathbf{y}) \Leftrightarrow \mathbf{x} \sim \mathbf{y}$. As $\pi_4(\mathbf{x})$ consists of infinite words, we have $\pi_4(\mathbf{x}) \notin T_+$, and π_4 is associative.

Let $\cong$ be the equivalence relation on A^ω defined by $u \cong v \Leftrightarrow sign(u) = sign(v)$ for $u,v \in A^\omega$. Each value of π_4 is an equivalence class of $\cong$. From Proposition 6 follows that, conversely, each equivalence class of $\cong$ is the value of $\pi_4(\mathbf{x})$ for some $\mathbf{x} \in T_+^{\mathbb{N}}$.

By Proposition 6, the ω-product $[x_1x_2x_3\ldots]$ is now an equivalence class of $\cong$ containing $x_1x_2x_3\ldots$, which is analogous to $[x]$ denoting the equivalence class of $\equiv$ containing x. This is even more than analogy: the extension of $\cong$ to finite words obtained by allowing $u \in A^\infty$ in (9) is identical to $\equiv$. The value of π_4 extends thus in a natural way the notion of finite trace, and is a good candidate for an infinite trace. It is, in fact, exactly the infinite trace introduced by Kwiatkowska [5,6] and Gastin [2,3]. According to these authors, infinite trace is an equivalence class of relation $\approx$ on A^ω defined by

$$u \approx v \;\Leftrightarrow\; (v \ll u \;\text{ and }\; u \ll v), \tag{11}$$

$$\text{where}\quad v \ll u \;\Leftrightarrow\; (\text{for every }\; y \leq v \;\text{ exists }\; x \leq u \;\text{ such that }\; [y] \leq [x])\,. \tag{12}$$

Proposition 7. *The values of $\pi_4(\mathbf{x})$ for $\mathbf{x} \in T_+^{\mathbb{N}}$ are exactly the infinite traces defined by (11)–(12).*

Proof. It is enough to show that $v \ll u \Leftrightarrow sign(v) \subseteq sign(u)$ for $u,v \in A^\omega$.

Suppose $v \ll u$ and consider any $[z] \in sign(v)$. By (9), we have $[z] \leq [y]$ for some $y \leq v$. By (12), exists $x \leq u$ such that $[y] \leq [x]$. We have $[z] \leq [y] \leq [x]$; from (9) follows $[z] \in sign(u)$.

Suppose now $sign(v) \subseteq sign(u)$ and take any $y \leq v$. We have $[y] \in sign(v) \subseteq sign(u)$, which means $[y] \leq [x]$ for some $x \leq u$; from (12) follows $v \ll u$. □

Let us return now to the sequence $\mathbf{x} = [ab],[ab],[ab],\ldots$ where $(a,b) \in I$. One can easily see that $sign(\mathbf{x})$ is the set of all traces $[a^mb^n]$ with arbitrarily large $m \geq 0$ and $n \geq 0$. It is the same as signature of any word consisting of infinitely many a's and b's. The signatures of words with only finitely many a's are different: they do not contain $[a^mb^n]$ with m above certain value. The same holds symmetrically for words with only finitely many b's. It follows that $\pi_4(\mathbf{x})$ is exactly the set of words with infinitely many a's and b's.

We wrap up by checking that $\pi_4(\mathbf{x})$ still induces $wsign(\mathbf{x})$ as the set of its prefixes:

Proposition 8. *For any $\mathbf{x} \in T_+^{\mathbb{N}}$, $\downarrow(\pi_4(\mathbf{x})) = wsign(\mathbf{x})$.*

Proof. Let $\mathbf{x} = [x_1],[x_2],[x_3],\ldots \in T_+^{\mathbb{N}}$. Take any $x \in\downarrow(\pi_4(\mathbf{x}))$, that is, $x \leq u$ for some $u \in \pi_4(\mathbf{x})$. By (9), $[x] \in sign(u)$, and by (10), $[x] \in sign(\mathbf{x})$; by (6), $x \in wsign(\mathbf{x})$.
Take now any $x \in wsign(\mathbf{x})$. By (6), $[x] \in sign(\mathbf{x})$, that is, $[x] \leq [x_1\ldots x_i]$ for some $i \geq 1$. This implies $x \leq z$ for some $z \in [x_1\ldots x_i]$. Denote $v = zx_{i+1}x_{i+2}x_{i+3}\ldots$. By Proposition 6, $sign(v) = sign([z],[x_{i+1}],[x_{i+2}],[x_{i+3}],\ldots)$. But $[z] = [x_1\ldots x_i]$; by Propositions 2 and 1 we have $sign(v) = sign(\mathbf{x})$, so $v \in \pi_4(x)$. By construction of v, we have $x \leq v$, so $x \in\downarrow(\pi_4(\mathbf{x}))$. □

6 Including null trace

Results from the preceding section do not extend smoothly to sequences including $[\varepsilon]$. The sequences ending with $[\varepsilon]^{\mathbb{N}} = [\varepsilon],[\varepsilon],[\varepsilon],\ldots$ deviate from the patterns established in Section 5. In addition, their products do not quite correspond to the notion of infinite trace. For the sake of completeness, we outline the consequences of allowing null trace.

Proposition 1 that was the foundation of our constructions for $\mathbf{T}$ remains valid for the semigroup $(T,\cdot)$ of all traces. The Lemmas that were used to prove it are valid for $(T,\cdot)$, with the only modification that we have to allow $[z_i] = [\varepsilon]$ in Lemma 3.

Definitions of π_1 and π_2 are applicable to all sequences in $T^{\mathbb{N}}$ and the resulting products satisfy (A1)–(A2). The result of $\pi_1([x_1],\ldots,[x_n],[\varepsilon]^{\mathbb{N}})$ is a finite set $\downarrow [x_1 \ldots x_n]$ of traces, and the result of $\pi_2([x_1],\ldots,[x_n],[\varepsilon]^{\mathbb{N}})$ is the corresponding finite set of words. In particular, we have $\pi_1([\varepsilon]^{\mathbb{N}}) = \{[\varepsilon]\}$ and $\pi_2([\varepsilon]^{\mathbb{N}}) = \{\varepsilon\}$. These results are not in T only if we keep a strict distinction between $\{[\varepsilon]\}$, $\{\varepsilon\}$, and $[\varepsilon]$. If we do not (which is quite common), they are in T, and do not satisfy (A3).

Definition (8) does not apply in a meaningful way to sequences ending with $[\varepsilon]^{\mathbb{N}}$. We may modify it to correctly specify the set of least upper bounds also for finite $wsgn(\mathbf{x})$. Or, what is equivalent, define $\pi_3([x_1],\ldots,[x_n],[\varepsilon]^{\mathbb{N}}) = [x_1 \ldots x_n]$ as a special case. This extension satisfies $\pi_3(\mathbf{x}) = \pi_3(\mathbf{y}) \Leftrightarrow \mathbf{x} \sim \mathbf{y}$, but the result $[x_1 \ldots x_n]$ is in T. However, we have

$$\pi_3([x],[x_1],\ldots,[x_n],[\varepsilon]^{\mathbb{N}}) = [x] \cdot [x_1 \ldots x_n]\,,$$

so this result satisfies (A3) and the extended π_3 is associative.

Definitions (9) and (10) are perfectly applicable to any $u \in A^{\infty}$. Proposition 6 must be adjusted by assuming a convention that $x\varepsilon\varepsilon\varepsilon\ldots = x$. We again have a finite trace $[x_1 \ldots x_n]$ as the result of $\pi_4([x_1],\ldots,[x_n],[\varepsilon]^{\mathbb{N}})$. As before, it satisfies (A3) and the extended π_4 is associative.

7 Conclusions

The fact that two known definitions of infinite trace coincide with associative ω-products indicates a strong relationship between these ideas.

We note that all four ω-products discussed in Section 5 are "free" in the sense defined in [13], and thus isomorphic. There is a one-to-one mapping between each pair that preserves the infinite product and the mixed product. In other words, all four are "the same" and differ only by the choice of values. Indeed, the values we chose for π_1–π_4 can be uniquely converted into each other: each of them uniquely defines $sign(\mathbf{x})$, from which one can reconstruct the sequence $\mathbf{x}$ – up to similarity – as shown in the proof of Proposition 3, part (2).

Our four ω-products may be regarded as different representations of the same abstract object. One is tempted to define infinite trace to be just a free associative ω-product of finite non-null traces, without bothering how it is represented. But, the choice of representation is significant when it comes to properties other than associativity. As we have seen, π_4 can be viewed as a nice extension of finite traces while π_3 has a disturbing counter-intuitive property.

According to common sense, infinite behavior involving fairness constraints (such as b having to occur infinitely often) cannot be adequately defined by means of finite initial sequences. It appears thus as a kind of paradox that correct infinite behavior, such as represented by values of π_4, is fully defined by a set of finite prefixes that is the value of π_2. It looks like the initial behavior together with the independence relation I is sufficient to represent fair behavior.

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