An improved construction of deterministic $\omega$-automaton from derivatives

Roman Redziejowski

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What is $\omega$-automaton?

Automaton: states, transitions

- Deterministic:
  - $S_0 \xrightarrow{b} S_0 \xleftarrow{a} S_1$
  - $S_0 \xrightarrow{a} S_1 \xleftarrow{b} S_0$

- Nondeterministic:
  - $S_0 \xrightarrow{a, b} S_1$
  - $S_0 \xrightarrow{a} S_1$

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$\omega$-automaton from derivatives
What is $\omega$-automaton?

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- Deterministic
  - $S_0$ to $S_1$: $a$, $b$
  - $S_1$ to $S_0$: $a$

- Nondeterministic
  - $S_0$ to $S_1$: $a$, $b$
  - $S_1$ to $S_0$: $a$

Omega-automaton: recognizes $\omega$-languages (sets of infinite words).
What is $\omega$-automaton?

Automaton: states, transitions

**Deterministic**

$\omega$-automaton: recognizes $\omega$-languages (sets of infinite words).

How: infinite word $w$ accepted $\iff$ exists an accepting run on $w$. 

**Nondeterministic**
What is $\omega$-automaton?

Automaton: states, transitions

Deterministic

$$
S_0 \xrightarrow{a} S_1 \xleftarrow{a}
$$

Nondeterministic

$$
S_0 \xrightarrow{a} S_1
$$

Omega-automaton:
recognizes $\omega$-languages (sets of infinite words).

How: infinite word $w$ accepted $\iff$ exists an accepting run on $w$.

Accepting run defined via set of states visited infinitely often
(Büchi, Muller, Rabin, Streett, parity...)

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$\omega$-automaton from derivatives
Accepting run can also be defined in terms of transitions.
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\[ G \text{ infinitely often } \& \ R \text{ finitely often recognizes } (a \cup b)^* (a^\omega \cup b^\omega). \]
Accepting run can also be defined in terms of transitions.

\( G \) infinitely often & \( R \) finitely often recognizes \((a \cup b)^*(a^\omega \cup b^\omega)\).

\( \text{Blob} \bullet \) infinitely often recognizes \((a \cup b)^*a^\omega\).
Each $\omega$-automaton recognizes an $\omega$-regular language described by an $\omega$-regular expression such as $(a \cup b)^*(a^\omega \cup b^\omega)$ or $(a \cup b)^*a^\omega$. 
Each $\omega$-automaton recognizes an $\omega$-regular language described by an $\omega$-regular expression such as $(a \cup b)^* (a^\omega \cup b^\omega)$ or $(a \cup b)^* a^\omega$.

Recalling: $\omega$-regular language is constructed from $\emptyset$, $\{\varepsilon\}$, and $\{a\}$ for $a \in \Sigma$ by a finite number of applications of union, product, star, omega.
Each $\omega$-automaton recognizes an $\omega$-regular language described by an $\omega$-regular expression such as 
\[(a \cup b)^* (a^\omega \cup b^\omega) \text{ or } (a \cup b)^* a^\omega.\]

Recalling: $\omega$-regular language is constructed from $\emptyset$, $\{\varepsilon\}$, and $\{a\}$ for $a \in \Sigma$ by a finite number of applications of union, product, star, omega.

($Regular$ language is constructed using only union, product, and star.)
Given an an $\omega$-regular expression construct deterministic $\omega$-automaton recognizing the language defined by that expression.
(Brzozowski 1964)

Derivative of $X \subseteq \Sigma^\infty$ with respect to $w \in \Sigma^*$: set of words obtained by stripping the initial $w$ from words in $X$ starting with $w$.

$$\partial_w X = \{z \in \Sigma^\infty \mid wz \in X\}$$
What is derivative?

(Brzozowski 1964)

Derivative of $X \subseteq \Sigma^\infty$ with respect to $w \in \Sigma^*$: set of words obtained by stripping the initial $w$ from words in $X$ starting with $w$.

$$\partial_w X = \{ z \in \Sigma^\infty \mid wz \in X \}$$

Use: suppose you check if input is in $X$. After reading $w$, remains to check if the rest is in $\partial_w X$. 
Results from Brzozowski 1964, extended to $\omega$-languages.

(1) An ($\omega$-)regular language has finitely many distinct derivatives.
Derivatives of $\omega$-regular language

Results from Brzozowski 1964, extended to $\omega$-languages.

(1) An ($\omega$-)regular language has finitely many distinct derivatives.

(2) These derivatives are also ($\omega$-)regular and can be effectively computed using rules such as these:

\[
\partial_a \emptyset = \partial_a \{\varepsilon\} = \emptyset, \quad \partial_a (X \cup Y) = \partial_a X \cup \partial_a Y,
\]
\[
\partial_a \{a\} = \varepsilon, \quad \partial_a (XY) = (\partial_a X)Y \cup \nu(X)(\partial_a Y),
\]
\[
\partial_w a X = \partial_a (\partial_w X), \quad \text{etc}.
\]
Using derivatives to recognize regular language

Identify states with languages they recognize.
Identify states with languages they recognize.

Suppose you start in state $D_0 = (a \cup b)^*a$.
If you read $a$, go to state $\partial_a X = (a \cup b)^*a \cup \varepsilon = D_1$.
If you read $b$, go to state $\partial_b X = (a \cup b)^*a = D_0$. 
Using derivatives to recognize regular language

Identify states with languages they recognize.

Suppose you start in state $D_0 = (a \cup b)^* a$. 
If you read $a$, go to state $\partial_a X = (a \cup b)^* a \cup \epsilon = D_1$. 
If you read $b$, go to state $\partial_b X = (a \cup b)^* a = D_0$.

From state $D_1$: 
If there is no more input, you are done because $\epsilon \in D_1$. 
If you read $a$, go to state $\partial_a D_1 = D_1$. 
If you read $b$, go to state $\partial_b D_1 = D_0$. 

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$\omega$-automaton from derivatives
Using derivatives to recognize regular language

Identify states with languages they recognize.

Suppose you start in state $D_0 = (a \cup b)^*a$.
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If you read $b$, go to state $\partial_b X = (a \cup b)^*a = D_0$.

From state $D_1$:
If there is no more input, you are done because $\varepsilon \in D_1$.
If you read $a$, go to state $\partial_a D_1 = D_1$.
If you read $b$, go to state $\partial_b D_1 = D_0$. 
Automaton recognizing a **regular** language $X$.

- **States:** distinct derivatives of $X$.
- **Initial state:** $\partial_\varepsilon X$.
- **Transitions:** $D \xrightarrow{a} \partial_a D$.
- **Final state:** any derivative containing $\varepsilon$. 
Does not work for $\omega$-regular language

$$X = (a \cup b)^* (a^\omega \cup (ab)^\omega)$$
Does not work for $\omega$-regular language

$X = (a \cup b)^*(a^\omega \cup (ab)^\omega)$

$\partial_a X = (a \cup b)^*(a^\omega \cup (ab)^\omega) = X$

$\partial_b X = (a \cup b)^*(a^\omega \cup (ab)^\omega) = X$
Does not work for $\omega$-regular language

$$X = (a \cup b)^*(a^\omega \cup (ab)^\omega)$$

$$\partial_a X = (a \cup b)^*(a^\omega \cup (ab)^\omega) = X$$

$$\partial_b X = (a \cup b)^*(a^\omega \cup (ab)^\omega) = X$$

Too few transitions to recognize $X$. 

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\(\omega\)-automaton from derivatives
Distinguish derivatives that bite the omega part:

Insert "marker" \# before the operand of each $\omega$. Take derivatives with respect to $a$ and $\#a$. 
Distinguish derivatives that bite the omega part:

Insert "marker" † before the operand of each ω. Take derivatives with respect to a and †a.

For example:

\[ X = (a \cup b)^* (a^\omega \cup (ab)^\omega) \]

\[ X' = (a \cup b)^* ((†a)^\omega \cup (†ab)^\omega) \]
Distinguish derivatives that bite the omega part:

Insert "marker" $\#$ before the operand of each $\omega$. Take derivatives with respect to $a$ and $\#a$.

For example:

$$X = (a \cup b)^* (a^{\omega} \cup (ab)^{\omega})$$

$$X' = (a \cup b)^* ((\#a)^{\omega} \cup (\#ab)^{\omega})$$

$$\partial_a X' = X'$$

$$\partial_{\#a} X' = (\#a)^{\omega} \cup b (\#ab)^{\omega}$$
New derivative automaton:

- **States:** nonempty derivatives of $X'$. 
- **Initial state:** $\partial_\varepsilon X'$. 
- **Transitions:**
  
  - $D \xrightarrow{a} \partial_a D$, 
  
  - $D \xrightarrow{a/\bullet} \partial_\#_a D$ (enters $\omega$-iteration). 
- **Accepting run:** infinitely many transitions with $\bullet$. 

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$\omega$-automaton from derivatives
$X' = (a \cup b)^*((\#a)^{\omega} \cup (\#ab)^{\omega})$
Derivative automaton

\[ X' = (a \cup b)^* \left( (\#a)^\omega \cup (\#ab)^\omega \right) \]

\[ D_0 = \partial \varepsilon X' = X'; \]
\[ D_1 = \partial_{\#a} X' = (\#a)^\omega \cup b (\#ab)^\omega; \]
\[ D_2 = \partial_{\#ab} X' = (\#ab)^\omega; \]
\[ D_3 = \partial_{\#a\#a} X' = (\#a)^\omega; \]
\[ D_4 = \partial_{\#ab\#a} X' = b (\#ab)^\omega. \]

\[ \partial_a D_0 = \partial_b D_0 = D_0; \]
\[ \partial_{\#a} D_0 = D_1; \]
\[ \partial_{\#b} D_1 = \partial_{\#b} D_4 = D_2; \]
\[ \partial_{\#a} D_1 = D_3; \]
\[ \partial_{\#a} D_2 = D_4. \]
$X' = (a \cup b)^* ((\#a)^\omega \cup (\#ab)^\omega)$

\[ D_0 = \partial_\varepsilon X' = X'; \]
\[ D_1 = \partial_{\#a} X' = (\#a)^\omega \cup b (\#ab)^\omega; \]
\[ D_2 = \partial_{\#ab} X' = (\#ab)^\omega; \]
\[ D_3 = \partial_{\#aa} X' = (\#a)^\omega; \]
\[ D_4 = \partial_{\#ab\#a} X' = b (\#ab)^\omega. \]

\[ \partial_a D_0 = \partial_b D_0 = D_0; \]
\[ \partial_{\#a} D_0 = D_1; \]
\[ \partial_b D_1 = \partial_b D_4 = D_2; \]
\[ \partial_{\#a} D_1 = D_3; \]
\[ \partial_{\#a} D_2 = D_4. \]
Derivative automaton

\[ \omega \text{-automaton from derivatives} \]
Has run with infinitely many $\bullet \iff$ input is in $(a \cup b)^* (a^\omega \cup (ab)^\omega)$. 

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$\omega$-automaton from derivatives
Has run with infinitely many $\bullet \iff$ input is in $(a \cup b)^*(a^\omega \cup (ab)^\omega)$.

But, it is nondeterministic.
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But, it is nondeterministic.

There exist determinization methods.
Determinization

Different ways to obtain states of deterministic automaton.
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Safra 1988 used trees built from the original states.
Determinization

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RR 1999 used annotations to run tree.
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Piterman 2007 used a numbering trick to improve Safra’s trees.
Different ways to obtain states of deterministic automaton.

Safra 1988 used trees built from the original states.

RR 1999 used annotations to run tree.

Piterman 2007 used a numbering trick to improve Safra’s trees.

We are going to improve RR 1999 by using annotations to run DAG\(^1\) enhanced with Piterman’s trick.

\(^1\)Directed Acyclic Graph
Run DAG

All possible runs on given input.

Input is in $X$ if and only if the DAG contains a *live path*: path with infinitely many $\bullet$.
Annotating the run DAG

input \{ D_0 \}

a \rightarrow \{ D_0 \} \rightarrow \{ D_0, D_1 \}

a \rightarrow D_0 \rightarrow D_1 \rightarrow D_3

b \rightarrow D_0 \rightarrow D_2

a \rightarrow D_0 \rightarrow D_1 \rightarrow D_4

... ... ...

Brackets enclose descendants + any node reached via •

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Annotating the run DAG

input \{ D_0 \}

a
\{ D_0 \} \{ D_1 \} \{ \{ D_3 \} \} \{ \{ \} \} \{ \} \{ \} \{ \} \{ \}

b
\{ D_0 \} \{ D_2 \}

a
\{ D_0 \} \{ D_1 \} \{ D_3 \}

\{ D_0 \} \{ D_1 \} \{ D_4 \}

Brackets enclose descendants + any node reached via •

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Easier to do it on the side...

Roman Redziejowski  \( \omega \)-automaton from derivatives
Watch for this situation:

\[
\begin{align*}
\{ D_{i_1} \ldots D_{i_n} \} & \quad \text{level } i \\
\ldots \ldots \\
\{ \{ D_{j_1} \} \{ D_{j_2} \} \ldots \{ D_{j_m} \} \} & \quad \text{level } j
\end{align*}
\]
Watch for this situation:

\[
\begin{aligned}
\{ D_{i_1}, \ldots, D_{i_n} \} & \quad \text{level } i \\
\ldots \ldots & \\
\{ \{ D_{j_1} \}, \{ D_{j_2} \}, \ldots, \{ D_{j_m} \} \} & \quad \text{level } j
\end{aligned}
\]

All paths from level \( i \) to level \( j \) are marked with •.
Watch for this situation:

\[
\begin{align*}
&\{ D_{i_1}, \ldots, D_{i_n} \} \quad \text{level } i \\
\Rightarrow \quad &\{ \{ D_{j_1} \} \{ D_{j_2} \} \ldots \{ D_{j_m} \} \} \quad \text{level } j
\end{align*}
\]

All paths from level \( i \) to level \( j \) are marked with \( \bullet \). We call this "green event" for the enclosing brackets, remove inner brackets, and emit green light.

\[
\begin{align*}
&\{ D_{i_1}, \ldots, D_{i_n} \} \\
\Rightarrow \quad &\{ D_{j_1}, D_{j_2}, \ldots, D_{j_m} \} \quad \Rightarrow \ G
\end{align*}
\]
Green event

Watch for this situation:

\[
\begin{align*}
\{ D_{i_1}, \ldots, D_{i_n} \} & \quad \text{level } i \\
\ldots \ldots & \\
\{ \{ D_{j_1} \}, \{ D_{j_2} \}, \ldots, \{ D_{j_m} \} \} & \quad \text{level } j
\end{align*}
\]

All paths from level \( i \) to level \( j \) are marked with \( \bullet \). We call this "green event" for the enclosing brackets, remove inner brackets, and emit green light.

\[
\begin{align*}
\{ D_{i_1}, \ldots, D_{i_n} \} \\
\ldots \ldots \\
\{ D_{j_1}, D_{j_2}, \ldots, D_{j_m} \} & \Rightarrow G
\end{align*}
\]

Repeated green events \( \Rightarrow \) live path exists.
Green event

input \{ D_0 \} → \{ D_0 \}

a

\{ D_0 \} \{ D_1 \} \{ D_3 \} → \{ D_0 \{ D_1 \} \{ D_3 \} \} → G

a

\{ D_0 \} \{ D_1 \} \{ D_2 \} \{ D_4 \} → \{ D_0 \{ D_1 \} \{ D_2 \} \{ D_4 \} \}

a

... ... ... ...

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ω-automaton from derivatives
It must be the same pair of brackets all the time!

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\[ \text{\( \omega \)-automaton from derivatives} \]
Solution: numbering

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\( \omega \)-automaton from derivatives
But numbers cannot grow to $\infty$ - must be reused

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$\omega$-automaton from derivatives
Red event to signal reuse: next G2 is another path

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ω-automaton from derivatives
Acceptance condition

Live path exists - that is, input is in $X$ - if and only if

$\Rightarrow$ $G2$ occurs infinitely often and

$\Rightarrow$ $R2$ occurs finitely often.
Acceptance condition

Live path exists - that is, input is in $X$ - if and only if

$\Rightarrow$ $G_2$ occurs infinitely often and

$\Rightarrow$ $R_2$ occurs finitely often.

(But just wait, it will be more complicated.)
Meanwhile, note this:

No pictures needed!

We can produce annotations without ever constructing the derivative automaton or drawing the DAG!
Meanwhile, note this:

No pictures needed!

We can produce annotations without ever constructing the derivative automaton or drawing the DAG!

Start with $\{ D_0 \}$.  

$\begin{array}{c}
1 \\
1 \\
\end{array}$  

$\omega$-automaton from derivatives
Meanwhile, note this:

No pictures needed!

We can produce annotations without ever constructing the derivative automaton or drawing the DAG!

Start with \( \{ D_0 \} \).

For input letter \( a \), just replace every occurrence of \( D_i \) by

\[
\partial_a D_i \{ \partial_{(\# a)} D_i \}
\]

then remove empty derivatives, remove empty brackets, add numbers (indicating reuse), and handle green events.
But there is a snag...

\[
\begin{align*}
\{ D_1 \} \quad \{ D_3 \} \\
3 & \quad 3 & \quad 2 & \quad 2
\end{align*}
\]

becomes

\[
\begin{align*}
\{ \{ D_3 \} \} \quad \{ \{ D_3 \} \} \\
3 & \quad 5 & \quad 5 & \quad 3 & \quad 2 & \quad 6 & \quad 6 & \quad 2
\end{align*}
\]
But there is a snag...

What to do here? Have to delete one of $D_3$'s. Which one? We may miss live path.
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RR 1999 uses left-to-right ordering and retains the rightmost.
But there is a snag...

What to do here? Have to delete one of $D_3$'s. Which one? We may miss live path.

Safra 1988 orders nodes by "age" and retains the "oldest" predecessor.

RR 1999 uses left-to-right ordering and retains the rightmost.

Piterman 2007 exploits the numbering. We are going to use his trick.
Part 1 of the trick is numbering and renumbering of brackets. New brackets get a number higher than those present. Removal of empty brackets may leave gaps in the numbering:

1 2 4 6

We close the gaps by reducing numbers above the gap:

Number 4 is changed to 3
Number 6 is changed to 4

1 2 4 6
↓ ↓ ↓ ↓
1 2 3 4
Part 2 of the trick is: from multiple occurrences of $D_i$ retain one with the lowest nesting pattern.

Nesting patterns for $D_3$ are (1 - 3 - 5) and (1 - 2 - 6). The second is lexicographically lower. We remove the first occurrence of $D_3$:
Summing up...

How to get the next annotation:

(A1) Replace each $D_i$ as described. Each time assign the lowest unused number to new brackets.

(A2) Remove duplicates, leaving one with lowest nesting pattern.

(A3) Remove all empty pairs of brackets.
   Set $r =$ the lowest number on removed pair or $n + 1$ if none removed ($n =$ number of derivatives).

(A4) Handle green events.
   Set $g =$ the lowest number on green pair or $n + 1$ if none.

(A5) Renumber brackets to fill the gaps.

(A6) If $g < r$, append $\Rightarrow G \ g$ on the right.
   If $r \leq g$ and $r \neq n + 1$, append $\Rightarrow R \ r$. 
Example for input a

before:
\[ \{ D_0 \{ D_1 \} \{ D_3 \} \} \]
\[
\begin{array}{cccc}
1 & 3 & 3 & 2 & 2 \\
\end{array}
\]

replace \(D_i\)'s:
\[ \{ D_0 \{ D_1 \} \{ \{ D_3 \} \} \{ \{ D_3 \} \} \} \]
\[
\begin{array}{cccccc}
1 & 4 & 4 & 3 & 5 & 5 & 3 & 2 & 6 & 6 & 2 & 1 \\
\end{array}
\]

remove duplicates:
\[ \{ D_0 \{ D_1 \} \{ \{ D_3 \} \} \{ \{ D_3 \} \} \} \]
\[
\begin{array}{cccccc}
1 & 4 & 4 & 3 & 5 & 5 & 3 & 2 & 6 & 6 & 2 & 1 \\
\end{array}
\]

remove empty brackets:
\[ \{ D_0 \{ D_1 \} \{ \{ D_3 \} \} \} \]
\[
\begin{array}{cccccc}
1 & 4 & 4 & 2 & 6 & 6 & 2 & 1 \\
\end{array}
\]

handle green events:
\[ \{ D_0 \{ D_1 \} \{ D_3 \} \} \]
\[
\begin{array}{cccc}
1 & 4 & 4 & 2 \\
\Rightarrow g = 2 \\
\end{array}
\]

renumber:
\[ \{ D_0 \{ D_1 \} \{ D_3 \} \} \]
\[
\begin{array}{cccc}
1 & 3 & 3 & 2 \\
\Rightarrow r = 3 \\
\end{array}
\]

add output:
\[ \{ D_0 \{ D_1 \} \{ D_3 \} \} \]
\[
\begin{array}{cccc}
1 & 3 & 3 & 2 \\
\Rightarrow G 2 \\
\end{array}
\]
Deterministic automaton

Only finitely many distinct annotations exist, so the following automaton will be finite:

- **States:** Annotations reachable from the initial state by transitions defined below.

- **Initial state:** \( \{ \partial \epsilon X' \} \).

- **Transitions:** For a state \( s \) and an input letter \( a \in \Sigma \), apply (A1)–(A6) to \( s \). The part of the result between, and including, the brackets numbered 1 is the next state. The output is to the right of \( \Rightarrow \) (if any).

- **Acceptance condition:** A word \( w \in \Sigma^\omega \) is accepted if and only if exists \( g \) such that the automaton applied to \( w \) emits \( Gg \) infinitely many times, and emits any \( Rr \) with \( r \leq g \) only finitely many times.
States & transitions for $X = (a \cup b)^* (a^\omega \cup (ab)^\omega)$

$A = \{ D_0 \} \quad \xrightarrow{a} B \quad \xrightarrow{b} A$

$B = \{ D_0 \{ D_1 \} \} \quad \xrightarrow{a} C \Rightarrow G2 \quad \xrightarrow{b} D$

$C = \{ D_0 \{ D_1 \} \{ D_3 \} \} \quad \xrightarrow{a} C \Rightarrow G2 \quad \xrightarrow{b} D \Rightarrow R2$

$D = \{ D_0 \{ D_2 \} \} \quad \xrightarrow{a} E \Rightarrow G2 \quad \xrightarrow{b} A \Rightarrow R2$

$E = \{ D_0 \{ D_1 \} \{ D_4 \} \} \quad \xrightarrow{a} C \Rightarrow R2 \quad \xrightarrow{b} D \Rightarrow R3$
Automaton for $X = (a \cup b)^* (a^\omega \cup (ab)^\omega)$

Accepting run:
G2 infinitely often, R2 finitely often.
Don’t care about R3.
Using the method of Safra / Piterman one can estimate the maximum number of possible states to $n^n(n - 1)!$ where $n =$ number of states of derivative automaton.

For $n = 5$ this gives 75000.

How come we got only 5 states?
That’s all folks ... 

Thanks for your attention!